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LETTER TO THE EDITOR

Asymptotic winding angle distributions for two-dimensional Levy flights

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Abstract. We study the winding angle Θ of planar Levy flights around a given point, say the origin O. In particular, we calculate the variance $\langle\theta^2\rangle$ and show that, after a long time t, it is proportional to $\ln t$ (for Brownian random walks $\langle\theta^2\rangle \propto (\ln t)^2$). In the same conditions, we show that the variable $x = \theta/\sqrt{\ln t}$ is approximately distributed according to a Gaussian law. Finally, we compare our results with computer simulations.

Over several decades, the study of winding properties of Brownian curves has aroused great interest among mathematicians [1-4] and physicists [5-8]. Consider for instance a planar Brownian motion starting at some point different from the origin O and call $\theta(t)$ the angle wound around O at time t. The asymptotic probability distribution of $\theta(t)$, when, $t \to \infty$, was first calculated by Spitzer [2] who obtained a Cauchy law:

$$P\left(X = \frac{2\theta}{\ln t}\right) = \frac{1}{\pi}\left(\frac{1}{1+X^2}\right) \qquad \langle\theta^2\rangle = +\infty. \tag{1}$$

Of course, for discrete random walks, $\langle\theta^2\rangle$ remains finite [3] ($\langle\theta^2\rangle \propto (\ln t)^2$, t being the number of steps). Since that time, many other important laws have been discovered. For instance, Pitman and Yor [4] calculated the joint law for the windings around n prescribed points.

More recently, Duplantier and Saleur [8] studied self-avoiding random walks (SAW), obtaining (when $t \to \infty$) the following Gaussian distribution:

$$P\left(X = \frac{\theta}{2\sqrt{\ln t}}\right) = \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-X^2}. \tag{2}$$

However, the situation appears less clear if we consider winding properties of more general Levy flights [10]. The purpose of this letter is the study of some of them essentially through the calculation of the variance $\langle\theta^2\rangle$ (in the limit $t \to \infty$).

We begin with the definition of two-dimensional isotropic Levy flights. The probability for a particle starting at $\boldsymbol{r}_0$ ($t = 0$) to reach $\boldsymbol{r}$ at time t reads [10]:

$$P(\boldsymbol{r}_0, \boldsymbol{r}; t) \equiv P(\boldsymbol{r} - \boldsymbol{r}_0; t) = \left(\frac{1}{2\pi}\right)^2 \int \mathrm{d}^2\boldsymbol{k}\, \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_0) - t|\boldsymbol{k}|^\mu} \tag{3}$$

where $|\boldsymbol{k}| = (k_x^2 + k_y^2)^{1/2}$ and $0 < \mu \leq 2$.

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In fact, due to isotropy, it only depends on the end-to-end distance $|\boldsymbol{r}-\boldsymbol{r}_0|$ that scales like $t^{1/\mu}$. Distribution (3) is stable and indefinitely divisible. It satisfies the Chapman–Kolmogorov equation:

$$P(\boldsymbol{r};\, t)=\int \mathrm{d}^2\boldsymbol{r}'\cdot P(\boldsymbol{r}-\boldsymbol{r}';\, t-\tau)P(\boldsymbol{r}';\, \tau) \qquad 0\leqslant\tau\leqslant t. \tag{4}$$

Notice that $P(\boldsymbol{r};\, t)$ can be evaluated in a closed form only in a few special cases:
(i) $\mu=2$ (Brownian random walks)

$$P(\boldsymbol{r};\, t)=\frac{1}{4\pi t}\,\mathrm{e}^{-r^2/4t} \tag{5}$$

satisfying the diffusion equation $\Delta P=\partial P/\partial t$.
(ii) $\mu=1$ (Cauchy flights)

$$\begin{aligned} P(\boldsymbol{r};\, t)&=\frac{1}{2\pi}\int_0^\infty \mathrm{d}k\, kJ_0(|\boldsymbol{k}|\cdot|\boldsymbol{r}|)\,\mathrm{e}^{-t|\boldsymbol{k}|} \\ &=\frac{1}{2\pi}\,\frac{t}{(t^2+r^2)^{3/2}} \end{aligned} \tag{6}$$

with $\Delta P=-\partial^2 P/\partial t^2$.

More generally, for $0<\mu<2$ and $r\to\infty$, we can obtain the following asymptotic development [10] ($\eta=tr^{-\mu}$):

$$\begin{aligned} P(\boldsymbol{r};\, t)&=\frac{1}{r^2}[C_\mu\eta+D_\mu\eta^2+\mathcal{O}(\eta^3)] \\ C_\mu&=\frac{2^\mu}{\pi^2}\left[\Gamma\left(\frac{\mu}{2}+1\right)\right]^2\sin\left(\frac{\pi\mu}{2}\right) \\ D_\mu&=-\frac{2^{2\mu-1}}{\pi^2}[\Gamma(\mu+1)]^2\sin(\pi\mu). \end{aligned} \tag{7}$$

Equations (7) show that, for $\alpha>\mu$, the moments $\langle r^\alpha\rangle$ are infinite: we are in the presence of broad distributions. They do not satisfy diffusion type equations. However, using (4) and (7), we can construct an integro-differential equation involving $\partial P(\boldsymbol{r};\, t)/\partial t$

$$\frac{\partial P(\boldsymbol{r};\, t)}{\partial t}=C_\mu\int \mathrm{d}^2\boldsymbol{r}'\,\frac{[P(\boldsymbol{r}';\, t)-P(\boldsymbol{r};\, t)]}{|\boldsymbol{r}-\boldsymbol{r}'|^{2+\mu}} \qquad 0<\mu<2 \tag{8}$$

Now, we consider the winding angle θ around O and calculate the variance $\langle\theta^2\rangle$ in a discretized version, each time step being equal to Δt. The notation will go as follows. The particle starts at $\boldsymbol{r}_0$ (r_0, $\theta(0)=0$) and arrives, at time t, at point $\boldsymbol{r}$ (r, $\theta(t)$) after a series of $(t/\Delta t)$ flights. (In the limit $t\to\infty$, we can take $r_0\simeq0$.) Between t and $(t+\Delta t)$, it flies along a straight line from $\boldsymbol{r}$ to $\boldsymbol{r}'$ (r', $\theta(t)+\Delta\theta(t)$). In those conditions, $\langle(\Delta\theta(t))^2\rangle$ reads:

$$\langle(\Delta\theta(t))^2\rangle=(2\pi)\int_{-\pi}^{+\pi}\mathrm{d}(\Delta\theta)\int_0^\infty \mathrm{d}r\int_0^\infty \mathrm{d}r'(\Delta\theta)^2 rr'\mathcal{P}(\boldsymbol{r};\, t)\mathcal{P}(\boldsymbol{r}'-\boldsymbol{r};\, \Delta t) \tag{9}$$

and, using symmetry, we can write:

$$\langle(\theta(t+\Delta t))^2\rangle=\langle(\theta(t))^2\rangle+\langle(\Delta\theta(t))^2\rangle. \tag{10}$$

Considering first the case $\mu=2$, the calculation leads to:

$$\langle(\Delta\theta(t))^2\rangle=\left(\frac{\Delta t}{t}\right)\frac{I'}{\pi} \tag{11}$$

where

$$I'=\int_0^{\pi}\mathrm{d}\varphi\int_0^{\pi/2}\mathrm{d}u\frac{\varphi^2\sin u}{(\sqrt{(\Delta t+t)/t}-\sin u\cos\varphi)^2}$$

t being large (Δt fixed), we get:

$$\begin{aligned}&I'\sim\frac{\pi}{2}\ln\left(\frac{t}{\Delta t}\right)\\&\langle(\Delta\theta(t))^2\rangle\sim\frac{1}{2}\left(\frac{\Delta t}{t}\right)\ln\left(\frac{t}{\Delta t}\right).\end{aligned} \tag{12}$$

Equation (10) implies:

$$\langle(\theta(t))^2\rangle\sim\left(\frac{1}{2}\ln\left(\frac{t}{\Delta t}\right)\right)^2\sim\left(\frac{1}{2}\ln t\right)^2 \tag{13}$$

when $t\to\infty$, a result already obtained by Berger [3].

So, the scaling variable $X\propto\theta/\ln t$ appears when $\mu=2$, the variance $\langle\theta^2\rangle$ remaining obviously finite as long as we work with the discretized version. However, when we go to the diffusion approximation (continuous version, $\Delta t\to0$), we see (equation (13)), that $\langle\theta^2\rangle$ becomes infinite, a property first deduced by Levy [1].

Now, before turning to the general case $\mu<2$, we look at $\mu=1$ (Cauchy flights) where explicit calculation can be done. After some algebra, (6) and (9) lead to:

$$\begin{aligned}&\langle(\Delta\theta(t))^2\rangle=\left(\frac{\Delta t}{t}\right)\frac{I''}{\pi}\\&I''=\int_0^{\pi}\mathrm{d}\varphi\int_0^{\pi/2}\mathrm{d}u\frac{\varphi^2\cos u}{\sqrt{\beta}((\Delta t/t)\sin u+\sqrt{\beta})^2}\\&\beta=1-\sin 2u\cos\varphi.\end{aligned} \tag{14}$$

Taking the limit $t\to\infty$:

$$I''\to I_1=\int_0^{\pi}\mathrm{d}\varphi\int_0^{\pi/2}\mathrm{d}u\frac{\varphi^2\cos u}{\beta^{3/2}}=\int_0^{\pi}\frac{\mathrm{d}\varphi\,\varphi^2}{1-\cos\varphi}=4\pi\ln 2. \tag{15}$$

Finally, we get (for Δt finite):

$$\begin{aligned}&\langle(\theta(t))^2\rangle^{1/2}\sim a_1\sqrt{\ln\left(\frac{t}{\Delta t}\right)}\sim a_1\sqrt{\ln t}\\&a_1=\sqrt{4\ln 2}\simeq1.665.\end{aligned} \tag{16}$$

Now, the scaling variable should be $X=\theta/\sqrt{\ln t}$. This is supported by computer simulations. We studied winding angle distributions for Cauchy flights of $t=200$, 400, 800, 1600 steps. ($\Delta t=1$; 10 000 events for each t value.) In particular, we obtained $\langle(\theta(t))^2\rangle^{1/2}=a\sqrt{\ln t}$ with $1.66<a<1.68$, a value close to a_1 (16).

In addition, we notice, (16), that the variance $\langle\theta^2\rangle$ becomes infinite in the continuous version ($\Delta t \to 0$).

The probability distribution $P(\theta)$ can be approximately determined by assuming:

$$\langle(\theta(t))^m(\Delta\theta(t))^n\rangle \simeq \langle(\theta(t))^m\rangle\langle(\Delta\theta(t))^n\rangle. \tag{17}$$

Keeping only leading-order terms, we get:

$$\frac{\partial\langle(\theta(t))^n\rangle}{\partial t} \simeq \frac{n(n-1)}{2}\langle(\theta(t))^{n-2}\rangle\frac{a_1^2}{t}. \tag{18}$$

Straightforward algebra leads to:

$$\langle e^{i\lambda\theta}\rangle \simeq t^{-a_1^2\lambda^2/2}$$

and

$$P\left(X = \frac{\theta}{\sqrt{\ln t}}\right) \simeq \frac{1}{\sqrt{2\pi a_1^2}} e^{-X^2/2a_1^2}. \tag{19}$$

In figure 1, a comparison is done between (19), (full curve), and computer simulations (closed circles: $t = 200$, squares: $t = 1600$; 10 000 events for each t value). An interesting agreement shows that (19) is really a good approximation. (In particular, the neglect of correlations (17), does not seem to be very important.)

In the general case ($\mu < 2$), scaling properties of (3) allow us to write (9) as:

$$\begin{aligned}\langle(\Delta\theta(t))^2\rangle &= (2\pi)\int_{-\pi}^{+\pi} d(\Delta\theta)\int_0^\infty dr\int_0^\infty dr'(\Delta\theta)^2 rr'\mathscr{P}(\boldsymbol{r};1)\mathscr{P}\left(\boldsymbol{r}'-\boldsymbol{r};\frac{\Delta t}{t}\right)\\ &= (2\pi)C_\mu\left(\frac{\Delta t}{t}\right)\int_{-\pi}^{+\pi} d(\Delta\theta)\int_0^\infty dr\int_0^\infty dr'(\Delta\theta)^2 rr'\frac{\mathscr{P}(\boldsymbol{r};1)}{|\boldsymbol{r}'-\boldsymbol{r}|^{2+\mu}}\end{aligned} \tag{20}$$

in the limit $t \to \infty$ (we used (7)).

Finally, we get the results:

$$\begin{aligned}\langle(\Delta\theta(t))^2\rangle &\sim \left(\frac{\Delta t}{t}\right)a_\mu^2\\ \langle(\theta(t))^2\rangle^{1/2} &\sim a_\mu\sqrt{\ln t}\end{aligned} \tag{21}$$

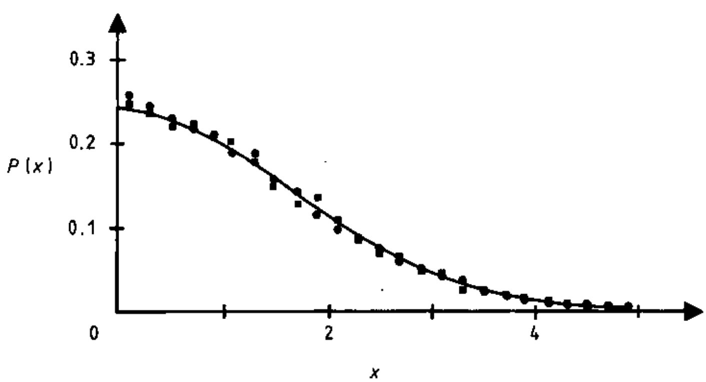

Figure 1. Computer simulations of Cauchy flights ($\mu = 1$). The winding angle distribution $P(X)$ is plotted as a function of the scaling variable $X = \theta/\sqrt{\ln t}$. ($P(X) = P(-X)$.) t is the number of steps (closed circles: $t = 200$; squares: $t = 1600$). The full curve represents the Gaussian, equation (19).

where

$$a_\mu = \sqrt{\frac{\mu I_\mu}{\pi}}$$
$$I_\mu = \int_0^\pi d\varphi \int_0^{\pi/2} du \frac{\varphi^2 (\sin u)^{\mu-1} \cos u}{(1-\sin 2u \cos\varphi)^{1+\mu/2}}. \tag{22}$$

(I_μ is finite when $0<\mu<2$. $I_1 = 4\pi \ln 2$), and for the winding angle distribution:

$$P\left(X = \frac{\theta}{\sqrt{\ln t}}\right) \simeq \frac{1}{\sqrt{2\pi a_\mu^2}} e^{X^2/2a_\mu^2}. \tag{23}$$

More generally, the qualitative considerations developed for $\mu = 1$ remain valid as long as $\mu < 2$. We stress that we have not found the analogue of Spitzer's law for Levy flights: we have always considered Δt finite.

Finally, we notice a striking similarity between our result (23) and that of Duplantier and Saleur (2) for the Brownian SAW. We think this fact is connected to another property shared by Levy flights and SAW: the end-to-end distance scales, for both, like t^α with $\alpha > 0.5$. Thus, the particle is, in average, far from O and the winding angle is reduced, compared to the free Brownian motion.

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